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Generalized momentum method to describe high-frequency solitary wave propagation in systems with varying dispersion

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We present a generalized momentum method to describe the evolution of the averaged (integral) pulse characteristics in nonlinear systems with periodically modulated dispersion and nonlinearity. A closed system of the ordinary differential equations is derived for averaged pulse power, width, and chirp. As a particular example, the developed theory is applied to the practical problem of the optical soliton transmission in dispersion-managed fiber links. [S1063-651X(98)51211-1]

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High-frequency pulse propagation in the nonlinear system with Kerr-like nonlinearity and periodically varying dispersion is governed by the nonlinear Schrödinger equation (NLSE) with periodic coefficients $d(z)$ and $c(z)$:

$$iA_z + d(z)A_{tt} + c(z)|A|^2A = 0. \tag{1}$$

This model is rather generic because it is derived under two very general assumptions: first, a high carrier frequency of the propagating wave packet and approximation of the dispersion curve near the carrier frequency by parabola [the second term in Eq. (1)], and second, the nonlinear part of the refractive index is assumed to be proportional to the intensity of the electric field (Kerr-type nonlinearity) [the third term in Eq. (1)]. These are very general and reasonable approximations in numerous physical applications. The NLSE has been derived in such different physical areas as plasmas, hydrodynamics, nonlinear optics, fiber optics, solid state physics, and many others (see, e.g., [1–6]). Therefore, we hope that the generalized momentum method to describe pulse evolution in Eq. (1) developed in this paper can find applications in a range of similar physical problems in which NLSE-based models occur. To be specific, we focus in this Rapid Communication on the optical applications of Eq. (1) (see, e.g., [1–6] and references therein). Then, normalized chromatic dispersion in Eq. (1) $d(z)=\tilde{d}(z)+\langle d\rangle$ represents the sum of a rapidly varying (over one compensation period) high local dispersion $\tilde{d}(z)$ and a constant residual dispersion $\langle d\rangle$ ($\langle d\rangle \ll \tilde{d}$, $\langle \tilde{d}\rangle = 0$); $c(z)$ accounts for power decay between amplifiers due to fiber loss. Angular brackets mean averaging over compensation period. Lumped action of the amplifiers is accounted for through transformation of the pulse power at junctions corresponding to locations of amplifiers. Equation (1) possesses conserved quantity $E=\int|A|^2dt$ that is the energy of the system.

The recently discovered dispersion-managed (DM) soliton [7,8] (or stretched pulse [9]) is a novel type of information carrier with properties [7–28] drastically different from that of a traditional fundamental soliton (soliton solution of the integrable NLSE). The advantage of the transmission of the soliton carrier signal is that it can be described by a few main parameters, such as pulse width, peak power, chirp parameter, and spectral width (the latter can be expressed through pulse width and chirp parameter). The particle-like behavior of the solitary wave signal allows us to make use of a well developed mathematical method to understand features of a such carrier and to predict effects occurring due to practical boundary conditions and due to deviations of real fiber properties from an ideal model. In the integrable and near-integrable models evolution of these few main soliton parameters can be calculated using perturbation methods [1–5,29]. In the general case some information can be gained by considering evolution of the integral quantities: different root-mean-square (RMS) momenta [1,30,31,12]. In this Rapid Communication we present a generalized momentum method to describe the main RMS DM soliton characteristics. The approach developed here is a generalization of the method suggested in our previous works [12,24,28]. This simple and transparent method is very useful in the modeling of an arbitrary dispersion-managed fiber links that typically involve many free parameters to be optimized.

To describe propagation dynamics of the main peak, let us consider, following [12] (see also papers [30,31], in which the momentum method has been used in other contexts), evolution of the integral (averaged over time) quantities related to the pulse width, RMS width T_{int}, and the integral pulse chirp M_{int},

$$T_{int}(z)=\left[\frac{\int t^2|A|^2dt}{\int|A|^2dt}\right]^{1/2},$$

$$T_{int}(z)M_{int}(z)=\frac{i}{4}\frac{\int t(AA_t^*-A^*A_t)dt}{\int|A|^2dt}. \tag{2}$$

Additional integral (averaged) pulse characteristics are root-mean-square pulse spectral bandwidth Ω_{RMS} and power P_{RMS}

$$\Omega_{RMS}^2(z)=\frac{\int\omega^2|A|^2d\omega}{\int|A|^2d\omega}=\frac{\int(|A|_t)^2dt}{\int|A|^2dt}+\frac{\int(\arg(A)_t)^2|A|^2dt}{\int|A|^2dt}$$

$$=\Omega_{mod}^2+\Omega_{phase}^2,$$

$$P_{RMS}(z)=\frac{\int|A|^4dt}{\int|A|^2dt}. \tag{3}$$

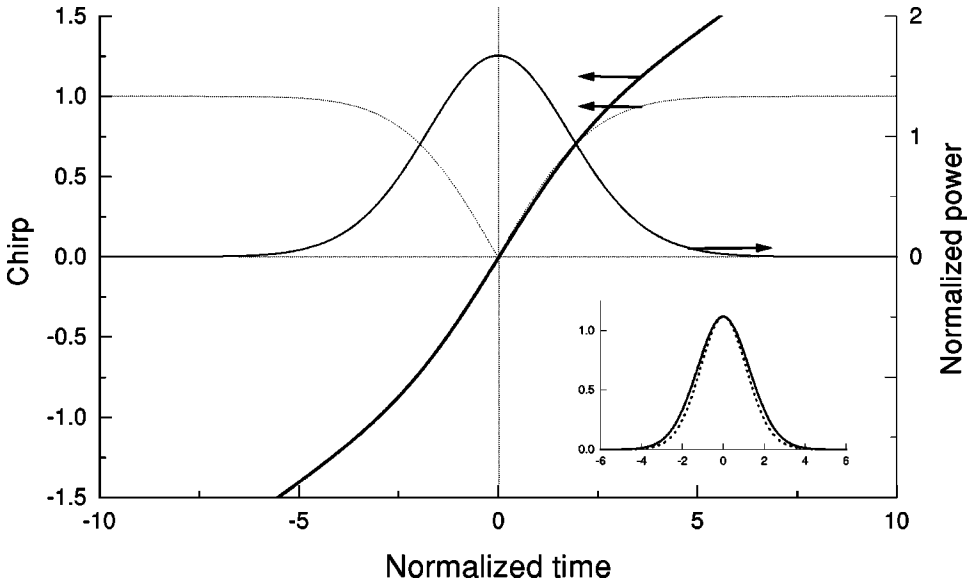

FIG. 1. First derivative over time of the pulse phase (pulse chirp) $\arg(A)_t$ (bold solid line) at $z=0.25$. Thick solid line is for the soliton power. Dashed line (function $|\int_{-t}^{+t}|A|^2dt/E|$) shows what part of the energy E is located in the interval from $-t$ to t. In the inset is plotted a function $|A|^4[\arg(A)_{tt}]$ (dotted line) from Eq. (6) shown at $z=0.25$ and the same, but with the parabolic approximation of the phase $2|A|^4\phi_2$ (solid line). Here and in the next figure $d(z)=\tilde{d}+\langle d\rangle=\pm 5+0.15$, $c(z)=1$.

Our goal is to derive (under minimal number of additional assumptions) a closed system of equations for the above RMS momenta. It is easy to check that the evolution of $T_{int}(z)$ and $M_{int}(z)$ is given by

$$\frac{dT_{int}}{dz}=4d(z)M_{int}(z), \tag{4}$$

$$\frac{d}{dz}(T_{int}M_{int})=d(z)\Omega_{RMS}^2-\frac{c(z)}{4}P_{RMS}. \tag{5}$$

To derive equations on Ω_{RMS} and P_{RMS} let us first introduce the following ancillary integral quantity $W_{int}(z)$,

$$W_{int}=i\frac{\int[(AA_t^*)^2-(A^*A_t)^2]dt}{\int|A|^2dt}=-\frac{\int|A|^4[\arg(A)]_{tt}dt}{\int|A|^2dt}. \tag{6}$$

Evolution equations on Ω_{RMS} and P_{RMS} then can be written in a simple form,

$$\frac{d}{dz}(\Omega_{RMS}^2)=c(z)W_{int},\quad \frac{d}{dz}P_{RMS}=2d(z)W_{int}. \tag{7}$$

The chirp (a first time derivative of the phase) of the typical DM pulses shows a linear behavior in the region where most of the energy is concentrated, as shown in Fig. 1. The bold solid line is for the first derivative of the phase over time (pulse chirp) $\arg(A)_t$ taken at $z=0.25$. The thick solid line is for the soliton power. The dashed line (function $|\int_{-t}^{+t}|A|^2dt/E|$) shows what part of the energy E is located in the interval from $-t$ to t. In the inset it is plotted a function $|A|^4[\arg(A)_{tt}]$ (dotted line) from Eq. (6) shown at $z=0.25$ and the same, but with the parabolic approximation of the phase $|A|^4[\arg(A)_{tt}]|_{t=0}=2|A|^4\phi_2$ (solid line). Of course, as is seen in Fig. 1, this is only a first approximation of the more complex phase picture and chirp is not linear at all in the whole time domain. However, note that the phase dependence appears in the above integral formulas only in the constructions like $|A|^4[\arg(A)]_{tt}$ being multiplied by $|A|^4$ or other powers {such as $|A|^2t[\arg(A)]_t$} of a fast decaying function $|A|^2$. Therefore, the contribution in the integral pulse characteristics due to deviations from the parabolic (in time) law in the phase is negligible in many practical situations (with highly localized $|A|^2$), as shown in the inset of Fig. 1. This is another justification of the use of a linear (in time) approximation for the DM pulse chirp (first derivative of the phase over time) in the energy-containing region (see [12,24]). Based on the above arguments, we now take a parabolic approximation (and a fourth-order term as a next-order correction) of the phase near the pulse peak power location $\arg(A)(z,t)=\Sigma_{n=0}^{\infty}\phi_{2n}(z)(t-t_0)^{2n}\approx\phi_0(z)+\phi_2(z)(t-t_0)^2+\phi_4(z)(t-t_0)^4+\cdots$ (here, t_0 is a position of the peak power, by additional transformation we always can set $t_0=0$; therefore, in what follows, $t_0=0$). Then we immediately recognize a simple relation between W_{int} and P_{int},

$$W_{int}(z)=-\sum_{n=1}^{\infty}2n(2n-1)\phi_{2n}K_{n-1}$$
$$\approx-2\phi_2(z)P_{RMS}(z)-12\phi_4(z)K_1+\cdots. \tag{8}$$

Here,

$$K_n=\frac{\int t^{2n}|A|^4dt}{\int|A|^2dt},\quad P_{RMS}=K_0.$$

As a first step, neglecting now terms with ϕ_4, we have five equations for the six quantities T_{int}, M_{int}, Ω_{RMS}, P_{RMS}, W_{int}, and ϕ_2. The missing last relation that is necessary to obtain a closed system of equations is given by

$$\frac{M_{int}(z)}{T_{int}(z)}=\frac{i}{4}\frac{\int t(AA_t^*-A^*A_t)dt}{\int t^2|A|^2dt}=\frac{\int|A|^2t[\arg(A)]_t dt}{2\int t^2|A|^2dt}$$

$$=\sum_{n=1}^{\infty} n\phi_{2n}S_n\approx\phi_2(z)+2\phi_4(z)S_2+\cdots,$$

$$S_n=\frac{\int t^{2n}|A|^2dt}{\int t^2|A|^2dt}. \tag{9}$$

Thus, in the leading order, the closed system of RMS momentum equations is presented by Eqs. (4), (5), (7)–(9) [with $\phi_4=0$ in Eqs. (8) and (9)].

Next we demonstrate that effectively the derived RMS equations can be transformed to the basic model developed in [12,24] (first obtained in context of the cascaded transmission lines by Gabitov and Turitsyn in [8] using variational approach). Thus, the fundamental model obtained in [8,12,24] presents the basic mathematical tool to optimize any dispersion-managed fiber links. First note that equations on P_{RMS} and Ω_{RMS} can be (after simple manipulations) integrated,

$$P_{RMS}(z)T_{int}(z)=P_{RMS}(0)T_{int}(0)=\text{const}_1, \tag{10}$$

$$\Omega_{RMS}^2(z)T_{int}^2(z)-4M_{int}^2(z)T_{int}^2(z)$$
$$=T_{int}^2(0)[\Omega_{RMS}^2(0)-4M_{int}^2(0)]=\text{const}_2. \tag{11}$$

As a particular result substituting now

$$\Omega_{RMS}^2(z)=4M_{int}^2(z)+\frac{\text{const}_2}{T_{int}^2(z)} \tag{12}$$

and Eq. (10) into Eq. (5) we get *exactly* the same RMS momentum equations on T_{int} and M_{int} as in [12,24,28],

$$\frac{dT_{int}}{dz}=4d(z)M_{int}(z),$$

$$\frac{d}{dz}M_{int}=\frac{d(z)\text{const}_2}{T_{int}^3}-\frac{c(z)\text{const}_1}{4T_{int}^2}. \tag{13}$$

This system of equations had been first obtained in the context of the optical signal propagation down the cascaded transmission systems by Gabitov and Turitsyn in [8] using variational approach and later in [12] using RMS momentum method. The advantage of the approach presented in this Rapid Communication is that we have used only *one* assumption about the structure (phase) of the DM pulse to derive these basic equations. Below we justify this assumption by direct numerical simulations. Here we also link to each other all important integral (averaged over time) characteristics of the optical pulse including RMS pulse width, chirp, power, and spectral bandwidth.

The natural generalization of this procedure will be to consider evolution of the higher-order K_n and S_n and to account for corresponding terms in the expansion of arg(A). Making use of higher order momentum quantities and assuming that ϕ_4 occurs as a small correction of the parabolic low in the phase (in terms of above discussion) we have a generalized equation for the RMS pulse characteristics and higher-order momenta. Equations (4), (5), (8), and (9) are the same [the latter two equations include now the terms with ϕ_4 (linear in ϕ_4)]. Equations (7) are modified as

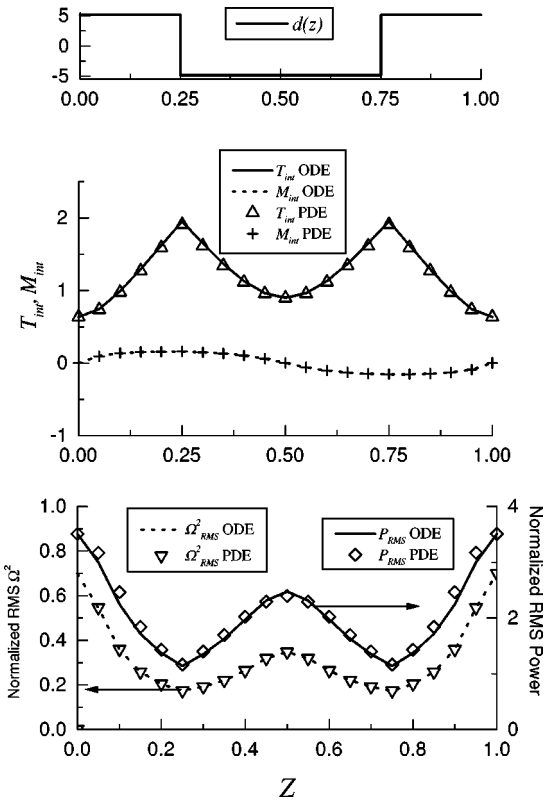

FIG. 2. Comparison of the RMS momentum (ODEs) model and direct simulations of Eq. (1). Dispersion map $d(z)=\pm5+0.15$ is plotted above. Middle figure shows evolution of T_{int} [solid line: ODEs model, triangles, Eq. (1)] and M_{int} (dashed line: ODEs RMS model and crosses, PDE (1)]. In the bottom figure it is shown evolution of the RMS pulse power P_{RMS} [solid line is found from ODE and the squares correspond to PDE (1)] and of the RMS spectral bandwidth Ω_{RMS}^2 [dashed line is for the solutions of ODEs and triangles are found from Eq. (1)].

$$\frac{d}{dz}(\Omega_{mod}^2)=-8d(z)\phi_2\Omega_{mod}^2+24d(z)\phi_4(1-2\Omega_2),$$

here

$$\Omega_2=\frac{\int t^2(|A|_t)^2dt}{\int|A|^2dt}, \tag{14}$$

$$\frac{d}{dz}(\Omega_{phase}^2)=8d(z)\phi_2\Omega_{mod}^2-2c(z)\phi_2P_{RMS}$$
$$+12\phi_4[4d(z)\Omega_2-2d(z)-c(z)K_1], \tag{15}$$

$$\frac{d}{dz}P_{RMS}=-4d(z)\phi_2P_{RMS}-24d(z)\phi_4K_1. \tag{16}$$

These equations are supplemented by the relations ($\phi_2\ll\phi_4$)

$$\Omega_{phase}^2=4\phi_2^2T_{int}^2+16\phi_2\phi_4S_2T_{int}^2 \tag{17}$$

and

$$\Omega_2 = \text{const}_3, \quad \frac{S_2}{T_{int}^2} = \text{const}_4, \quad \frac{K_1}{T_{int}} = \text{const}_5. \qquad (18)$$

The latter three relations have been obtained by integration (in the leading order in $\phi_4 \ll \phi_2$) of the equations for the higher-order momentum. The derived system of equations presents a generalization of the basic equations (13) accounting for the deviations from the parabolic approximation of the phase near the pulse peak.

Now we verify analytical results by numerical modeling. In Fig. 2 a comparison is shown of the evolution of the RMS pulse characteristics found by direct solving of Eq. (1) and by solving RMS ordinary differential equations (ODEs) as described above. True DM soliton dynamics is illustrated for the lossless model $[c(z)=1]$ and two-step map: $d(z)=d+\langle d\rangle$ if $0<z<L_1$ and $d(z)=-d+\langle d\rangle$ if $L_1<z<L$. The dynamics shown in Fig. 2 is for $d=5$, $\langle d\rangle=0.15$. A good enough agreement between ODEs consideration and direct simulations of Eq. (1) confirms validity of the assumption made above. Further improvement can be achieved considering next-order corrections.

Note that additional support to the above consideration can be found in recent publications [26,28,27], where it has been developed an advantageous expansion of an arbitrary DM pulse in the basis of chirped Gauss-Hermite functions. In a number of practical maps only a few terms (with the dominating role of the self-similar zero mode) in such an expansion are required to describe most of the important properties of the DM soliton. Therefore, as a first step, the phase of the DM soliton can be approximated in the central energy-containing part by the parabolic law (in time). This explains why the above method works so well. Equation (10) manifests a self-similar relation between pulse power and width.

In conclusion, we have presented a generalized RMS momentum equations method to describe an optical pulse propagating in dispersion-managed systems. Our approach is based only on the one assumption of the parabolic approximation of the phase of DM pulse in the central energy-bearing part of solution. This assumption is justified by numerical simulations showing an excellent agreement between simple RMS method consideration and direct simulations. The simply formulated RMS momentum method can be very useful for the massive numerical simulations required to optimize dispersion-managed fiber links.

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